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*All documents authorized.*  
*All answers must be justified.*

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## 1 Graph homomorphisms

Let  $D$  denote the graph of a *diamond* – the graph obtained from  $K_4$  by removing an edge.

- 1.1) Prove that  $G \rightarrow D$  if and only if  $G \rightarrow K_3$ .
- 1.2) Find a characterisation of graphs such that  $D \rightarrow G$ .

## 2 Outerplanar graphs

- 2.1) Recall the definition of an outerplanar graph.
- 2.2) Prove that every outerplanar graph contains a vertex of degree (at most) 2.
- 2.3) Prove that the treewidth of an outerplanar graph is (at most) 2.
- 2.4) Prove that every outerplanar graph is 3-choosable.
- 2.5) Is it true that every 2-degenerate graph has treewidth at most 2?

## 3 Random graphs

We would like to generate a random tree. We want to show that the Erdős-Rényi model  $\mathcal{G}(n, p)$  (where every edge appears with probability  $p$  independently from the others) is not a good model for that.

- 3.1) For a fixed  $p$ , what is the expected number of edges of  $\mathcal{G}(n, p)$ ? For which value of  $p$  (depending on  $n$ ) this expected number corresponds to the number of edges of a tree?

**3.2)** Let  $p_0$  be the answer to the previous question. Let  $X_i$  be the binary random variable indicating whether the vertex  $v_i$  is an isolated one in  $\mathcal{G}(n, p_0)$ ; let  $X$  denote the number of isolated vertices in  $\mathcal{G}(n, p_0)$ . Calculate the expectation of  $X$  (by first expliciting  $E(X_i)$ ). What is the asymptotics of  $E(X)$ ?

**3.3)** What can we say about the probability that  $\mathcal{G}(n, p_0)$  is a tree (knowing that for  $n > 1$  a tree has no isolated vertices)?

## 4 Vertex arboricity

The vertex arboricity of a graph  $G = (V, E)$  is the minimum number  $k$  of sets  $V_1, \dots, V_k$  such that  $V = V_1 \cup V_2 \cdots \cup V_k$ , and for all integer  $1 \leq i \leq k$ , the induced subgraph of  $G$  with vertex set  $V_i$  is a forest (i.e. has no cycle).

**4.1)** Fix an integer  $t$ . Can you find an upper bound on the highest possible value of the vertex arboricity of a graph  $G$  with  $\chi(G) = t$ , in terms of  $t$  (and not in terms of  $n$ )?

**4.2)** Prove that there exists a constant  $C$  such that every planar graph has vertex arboricity at most  $C$ . We do not ask for the best possible such constant.

**4.3)** What is the vertex arboricity of the complete graph  $K_n$  in terms of  $n$ ?

**4.4)** Fix an integer  $t$ . What is the lowest possible value of the vertex arboricity of a graph  $G$  with  $\chi(G) = t$ ?

**4.5)** Prove that every planar graph has vertex arboricity at most 3.